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Fluctuations of correlation functions in disordered spin systems

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Abstract. We study the fluctuations of the two-point correlation function in one-dimensional disordered spin models. These survive even in the thermodynamic limit and, in order to reconstruct their probability distribution from the moments, we study a set of generalised correlation lengths ζ_q. These moments may also be calculated within the transfer matrix formalism and provide insight on disorder-induced fluctuations.

We show that the ζ_q can be computed in Monte Carlo simulations. We discuss the crossover of the correlation decay rate at large distances to dominance by the most probable value given by ζ_0, and the relation with the finite-volume fluctuations of the free energy.

Finally we sketch how to extend our arguments to dimensions two and three.

1. Introduction

It is well known that in disordered systems such as spin glasses, one can have strong fluctuations from sample to sample so that average values can be very different from the most probable ones. This is the case for the partition function Z_N of an N-spin system. In the thermodynamic limit $N \to \infty$ the Z probability distribution becomes sharply peaked around $\exp(\overline{\ln Z_N})$ and this typically differs from the average partition function $\overline{Z_N}$. For this reason in disordered systems, one has to average the free energy (quenched average) instead of the partition function (annealed average). Throughout this paper, the bar indicates an average over the disorder realisation and the angular brackets, $\langle\ \rangle$, indicate a thermal average at fixed realisation of disorder. When $N \to \infty$ almost all samples have the same free energy $F = -\beta\, \overline{\ln Z}$. In the context of the transfer matrix formalism this is a rigorous result known as Oseledec's theorem [1]. On the other hand, the finite-volume fluctuations of the free energy can be characterised by the moments of the partition function $\overline{Z^q}$, as widely discussed in the literature [2-4]¶.

The description of fluctuations is most important for the two-point connected correlation function $G_i(r) = \langle \sigma_i \sigma_{i+r} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+r} \rangle$ between a spin σ_i and another spin at distance r from the site i. Although $G_i(r)$ as well as all its moments are self-averaging,

¶ A general discussion is presented in section 4.4 of [3]. The 1D Ising chain with random couplings and zero external field is discussed in [4].

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i.e. $\lim_{N\to\infty} N^{-1}\Sigma_i G_i(r)^q = \overline{G(r)^q}$, a moment of reflection shows that, in general situations, the probability distribution of correlation functions is non-trivial even in the thermodynamic limit. Indeed, even if it becomes sharply peaked for large r, $G_i(r)$ fluctuates wildly among samples or in a given sample among different sites i, also in the limit $N\to\infty$. The purpose of this paper is to characterise these fluctuations, and provide evidence that they are easily observable in an experimental study. Let us also stress that they are of great physical relevance in contrast to the free energy fluctuations which are just a finite-volume effect.

This point was originally raised for the spin–spin correlation function of one-dimensional disordered magnets by Derrida and Hilorst [5]. In the following we shall generalise their results, using techniques developed in the context of dynamical systems.

To set the stage, let us briefly discuss a simple example which exhibits the basic features of the general case: a random Ising chain at zero external magnetic field, with couplings J_i which are independent random variables chosen from a distribution $\rho(J_i)$. The Hamiltonian is

$$H = -\sum_{i=1}^{N} J_i\sigma_i\sigma_{i+1} \qquad \sigma_i = \pm 1 \tag{1.1}$$

where N is the number of spins and periodic boundary conditions, $\sigma_{N+i} = \sigma_i$, are assumed.

It is trivial to see that, for a fixed realization of J_i, the two-point correlation function is given by

$$\langle\sigma_i\sigma_{i+r}\rangle = \prod_{j=i}^{i+r-1} \tanh(\beta J_j). \tag{1.2}$$

From (1.2), since all J_i are independent, one has

$$\overline{\langle\sigma_i\sigma_{i+r}\rangle} = (\overline{\tanh \beta J})^r. \tag{1.3}$$

On the other hand (we assume—for simplicity—that all J_i are positive) in the limit of large r we can use the central limit theorem. Equation (1.2) implies that, for $r \gg 1$, the probability distribution of $\langle\sigma_i\sigma_{i+r}\rangle$ is close to a log-normal peaked around the most probable value

$$\langle\sigma_i\sigma_{i+r}\rangle_{\mathrm{mp}} = \exp[r\,\overline{\ln(\tanh \beta J)}]. \tag{1.4}$$

Note that when r is large $\langle\sigma_i\sigma_{i+r}\rangle_{\mathrm{mp}} \ll \overline{\langle\sigma_i\sigma_{i+r}\rangle}$, since the leading contribution to $\overline{\langle\sigma_i\sigma_{i+r}\rangle}$ comes from very rare (and correlated) events. As we have discussed, large sample-to-sample fluctuations survive in the thermodynamic limit (at least for finite r) and the entire probability distribution of the two-point correlation function has to be studied. Let us therefore introduce the moments of these fluctuations, the generalised correlation functions [5]

$$C_q(r) = (\overline{|\langle\sigma_i\sigma_{i+r}\rangle|^q})^{1/q}. \tag{1.5}$$

We expect that in absence of phase transition phenomena, these decay exponentially with characteristic correlation lengths ζ_q, i.e.

$$C_q(r) \sim \exp(-r/\zeta_q) \qquad \text{for large } r \tag{1.6}$$

where the ζ_q have a non-trivial q dependence

$$\zeta_q^{-1} = -q^{-1}\,\overline{\ln(\tanh \beta J)^q} \tag{1.7}$$

because of the contribution of large fluctuations. Let us stress that although log-normal is a good approximation, its tail cannot be trusted, and so its moments can be very different from the actual ζ_q, when q is not small. A natural framework for computing the set of correlation length ζ_q is the transfer matrix formalism. In disordered systems, this leads to an (infinite) product of random matrices, the trace of which is the partition function. The largest eigenvalue of the product (whose logarithm is the maximum Lyapunov exponent) is related to the free energy, while the ratio of the two largest eigenvalues is related to the correlation decay. The probability distribution of the finite-volume fluctuations of free energy can then be characterised by the so-called generalised Lyapunov exponents $L(q)$ (see [3] and references therein). On the other hand, it is possible to relate the ζ_q to the finite-N fluctuations of the difference between the first and the second Lyapunov exponents. The situation is therefore more complicated and only in particular cases one can relate ζ_q and $L(q)$ in a simple way.

In section 2 we study the fluctuations of free energy and correlation functions within the transfer matrix formalism and discuss their relation with the fluctuations of effective Lyapunov exponents. In section 3 we compare the results of extensive Monte Carlo simulations with the theoretical predictions of section 2. In section 4, we present the conclusion and discuss the possibility of extending our arguments to higher dimensionality.

2. Generalised scaling exponents and correlation functions

In this section we first briefly recall the definition of the generalised Lyapunov exponents [6] for products of random matrices and their relation to the fluctuations of the free energy in random magnetic systems [3, 4]. We then extend the formalism to include the correlation functions and their fluctuations.

Let us consider a random Ising chain with Hamiltonian:

$$H = -\sum_{i=1}^{N} (J_i \sigma_i \sigma_{i+1} + h_i \sigma_i) \qquad \sigma_i = \pm 1. \tag{2.1}$$

We use periodic boundary conditions $\sigma_{N+1} = \sigma_1$, and the couplings J_i and/or external magnetic field h_i are independent random variables. This model can be studied in terms of the product of the random transfer matrices

$$\mathbf{L}_i = \begin{pmatrix} \exp[\beta(J_i + h_i)] & \exp[\beta(-J_i + h_i)] \\ \exp[-\beta(J_i + h_i)] & \exp[-\beta(-J_i + h_i)] \end{pmatrix}. \tag{2.2}$$

The free energy per spin f_N is given by

$$f_N = -\frac{1}{\beta N} \ln Z_N \tag{2.3a}$$

$$Z_N = \mathrm{Tr}\left(\prod_{i=1}^{N} \mathbf{L}_i \right). \tag{2.3b}$$

Oseledec's theorem [1] ensures that, for almost all realisations of $\mathbf{L}_i$, the limit $N \to \infty$ of f_N exists, is unique and is given by

$$\lim_{N \to \infty} f_N = -\lambda_1 / \beta \tag{2.4}$$

where λ_1 is the maximal Lyapunov exponent of the product $\mathbf{A}_N = \prod_{i=1}^{N} \mathbf{L}_i$. The exponent λ_1 can be related to the rate of increase of a generic vector $\boldsymbol{\xi}(0) \in \mathbb{R}^2$

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \ln R^{(1)}(N) \tag{2.5a}$$

where

$$R^{(1)}(N) = \|\boldsymbol{\xi}(N)\| / \|\boldsymbol{\xi}(0)\| \qquad \boldsymbol{\xi}(i+1) = \mathbf{L}_i \boldsymbol{\xi}(i). \tag{2.5b}$$

In general, for a $d \times d$ matrix $\mathbf{A}_N$, we can consider the set of all d eigenvalues $\eta_k(N)$ with $k = 1, 2, \ldots, d$, of the matrix $\mathbf{A}_N^{*}\mathbf{A}_N$. The characteristic Lyapunov exponents λ_i are then defined as

$$\lambda_k = \lim_{N\to\infty} \frac{1}{2N} \ln \eta_k(N). \tag{2.6}$$

The characteristic Lyapunov exponents λ_k are non-random quantities, in the sense that, for almost all realisations of $\mathbf{L}_i$, they do not depend on the particular realisation. There are, nevertheless, finite-N fluctuations around their asymptotic values λ_k. In particular, the probability $P_N(\gamma_1)\,\mathrm{d}\gamma_1$ that, for a system of size N, the (effective) maximum Lyapunov exponent assumes a value between γ_1 and $\gamma_1 + \mathrm{d}\gamma_1$ can be reconstructed through the generalised Lyapunov exponents $L(q)$:

$$L(q) = \lim_{N\to\infty} \frac{1}{N} \ln \overline{[R^{(1)}(N)]^q}. \tag{2.7}$$

In fact, assuming that for $\gamma_1 \neq \lambda_1$ $P_N(\gamma_1)$ decays as $\exp[-NS(\gamma_1)]$, where $S(\gamma_1) > 0$ is called the Lyapunov spectrum, its relation with $L(q)$ is given by the Legendre transformation [7]

$$L(q) = \max_{\gamma_1} [q\gamma_1 - S(\gamma_1)]. \tag{2.8a}$$

Note that

$$\left.\frac{\mathrm{d}L(q)}{\mathrm{d}q}\right|_{q=0} = \lambda_1 \qquad S(\lambda_1) = 0. \tag{2.8b}$$

From the Jensen inequality one can show that $L(q)/q$ is a non-decreasing function of q. A linear behaviour, $L(q) = q\lambda_1$, indicates negligible fluctuations $O(1/N)$, instead of the standard $O(N^{-1/2})$ scaling expected by central limit theorem arguments [2].

Except for very particular disorder distributions, an analytic calculation of the maximum Lyapunov exponent, and thus of $L(q)$, is not at all straightforward. The interested reader is referred to [8] for perturbative calculations and to [9] for 'microcanonical' estimates.

This approach to free energy fluctuations can be extended to correlation functions. At fixed realisation of disorder the two-point correlation function can be written as

$$\langle \sigma_1 \sigma_{1+r} \rangle = \frac{\sum_{\sigma_1,\sigma_{1+r}} \sigma_1 \langle \sigma_1 | \mathbf{A}_r | \sigma_{1+r} \rangle \sigma_{1+r} \langle \sigma_{1+r} | \mathbf{A}_{N-r} | \sigma_1 \rangle}{\sum_{\sigma_1,\sigma_{1+r}} \langle \sigma_1 | \mathbf{A}_r | \sigma_{1+r} \rangle \langle \sigma_{1+r} | \mathbf{A}_{N-r} | \sigma_1 \rangle} \tag{2.9}$$

where

$$\mathbf{A}_r = \prod_{i=1}^{r} \mathbf{L}_i \qquad \mathbf{A}_{N-r} = \prod_{i=r+1}^{N} \mathbf{L}_i$$

and

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Denoting by $\exp[n\gamma_k(n)]$ $(k = 1, 2)$ the eigenvalues of the matrix $\mathbf{A}_n$, we can write $\mathbf{A}_n = \mathbf{C}_n^{-1}\tilde{\mathbf{A}}_n\mathbf{C}_n$, where

$$\tilde{\mathbf{A}}_n = \begin{pmatrix} \exp[n\gamma_1(n)] & 0 \\ 0 & \exp[n\gamma_2(n)] \end{pmatrix}.$$

The exponents γ_k can be ordered so that $|\gamma_1(n)| > |\gamma_2(n)|$ for all n. By definition, $\gamma_k(n)$ is the kth effective Lyapunov exponent of the product of n matrices $\mathbf{L}_i$, and $\lambda_k = \lim_{n\to\infty} \mathrm{Re}\ \gamma_k$. In the following, for the sake of simplicity, we limit ourselves to the case of real γ_k.

By using the sum rule $\Sigma_\sigma |\sigma\rangle\langle\sigma| = \mathbf{1}$, (2.9) becomes

$$\langle\sigma_1\sigma_{1+r}\rangle = \frac{\Sigma_{k,k'} C_{k,k'}^{(r,N-r)} \exp[r\gamma_k(r) + (N-r)\gamma_{k'}(N-r)]}{\Sigma_{k,k'} M_{k,k'}^{(r,N-r)} \exp[r\gamma_k(r) + (N-r)\gamma_{k'}(N-r)]} \tag{2.10}$$

with

$$C_{k,k'}^{(r,N-r)} = \sum_{\sigma_1} \langle S_k|\mathbf{C}_{N-r}|\sigma_1\rangle\sigma_1\langle\sigma_1|\mathbf{C}_r^{-1}|S_{k'}\rangle \sum_{\sigma_{1+r}} \langle S_k|\mathbf{C}_r|\sigma_{1+r}\rangle\sigma_{1+r}\langle\sigma_{1+r}|\mathbf{C}_{N-r}^{-1}|S_{k'}\rangle \tag{2.11}$$

$$M_{k,k'}^{(r,N-r)} = \langle S_k|\mathbf{C}_{N-r}\mathbf{C}_r^{-1}|S_{k'}\rangle\langle S_k|\mathbf{C}_r\mathbf{C}_{N-r}^{-1}|S_{k'}\rangle \tag{2.12}$$

where $|S_1\rangle = |1\rangle$ and $|S_2\rangle = |-1\rangle$.

As $\gamma_1(n) > \gamma_2(n)$ for all n (Perron-Frobenius theorem), one can easily compute $\langle\sigma_i\sigma_{i+r}\rangle$, for large r, by taking into account just the first two terms in the numerator of (2.10), i.e. $(k, k') = (1, 1)$ and $(2, 1)$, and the leading term in the denominator, i.e. $(k, k') = (1, 1)$. It follows that

$$G_i(r) = \langle\sigma_i\sigma_{i+r}\rangle - \langle\sigma_i\rangle\langle\sigma_{i+r}\rangle \sim \exp\{-[\gamma_1(r) - \gamma_2(r)]r\}. \tag{2.13}$$

As for the maximal Lyapunov exponent, the probability distribution of the fluctuations may be reconstructed by introducing another infinite set of correlation lengths ζ_q by

$$C_q(r) = (\overline{|G_i(r)|^q})^{1/q} \sim \exp(-r/\zeta_q) \qquad \text{large } r. \tag{2.14}$$

By analogy with the $L(q)$, the ζ_q are related to the sample-to-sample fluctuations of $\gamma_1(r) - \gamma_2(r)$ by

$$\overline{\exp\{-qr[\gamma_1(r) - \gamma_2(r)]\}} \sim \exp(-qr/\zeta_q) \qquad \text{large } r. \tag{2.15}$$

The probability distribution of the fluctuations of $\gamma_1(r) - \gamma_2(r)$ can be obtained from ζ_q through a Legendre transformation similar to (2.8*a*). Moreover, one immediately sees that the most probable correlation decay is selected by the limit $q \to 0$, i.e. by the 'quenched' average of the logarithm [cf (2.8*b*)]

$$\zeta_0^{-1} = \left.\frac{\mathrm{d}(q\zeta_q^{-1})}{\mathrm{d}q}\right|_{q=0} = \lambda_1 - \lambda_2.$$

An experimentalist usually measures $C_1(r)$, i.e. the spatial average of $G_i(r)$, and so finds ζ_1. Note that from the convexity of q/ζ_q, one has $\zeta_0 \leqslant \zeta_1$, i.e. the typical correlation decay is faster than that estimated by the average correlation.

The physically relevant question, however, is what happens to the correlation function C_1 at 'intermediate' values of r, where we expect to observe a crossover between the average behaviour and the most probable behaviour which must be recovered at very large values of r. We shall discuss this point in the framework of the numerical experiments described in the next section. Another important remark is that the fluctuations of the difference between effective Lyapunov exponents are not simply related to the fluctuations of the effective maximum Lyapunov exponent, and so the ζ_q to the $L(q)$.

Let us, however, describe an exception to the general situation, i.e. the one-dimensional random field Ising model (the J_i are fixed, the h_i random): a simple calculation shows that the determinant of $\mathbf{L}_i$ is constant, so that for all r the sum of the effective Lyapunov exponents $\gamma_1(r)+\gamma_2(r)$ is equal to

$$\lambda_1+\lambda_2=\ln|\det \mathbf{L}_i|=\Delta. \tag{2.16}$$

This relation can be used to eliminate $\gamma_2(r)$ in (2.13), yielding

$$G_i(r)\sim\exp\{r[\Delta-2\gamma_1(r)]\} \tag{2.17}$$

which, substituted into (2.14), leads to

$$\overline{\exp\{-qr[\gamma_1(r)-\gamma_2(r)]\}}\sim\exp\{[q\Delta+L(-2q)]r\} \tag{2.18}$$

so that

$$\zeta_q^{-1}=-\Delta-L(-2q)/q. \tag{2.19}$$

Nevertheless, even if in the general case the ζ_q are not simply related to the $L(q)$, one can calculate them directly. The procedure goes as follows. Use the relations

$$\prod_{i=1}^{r}\det \mathbf{L}_i=\exp\{[\gamma_1(r)+\gamma_2(r)]r\} \tag{2.20}$$

and

$$R^{(1)}(r)=\frac{\|\boldsymbol{\xi}(r)\|}{\|\boldsymbol{\xi}(0)\|}\sim\exp[r\gamma_1(r)] \tag{2.21}$$

where $\boldsymbol{\xi}(i+1)=\mathbf{L}_i\boldsymbol{\xi}(i)$, so that:

$$\exp\{-[\gamma_1(r)-\gamma_2(r)]r\}\sim\frac{\prod_{i=1}^{r}\det \mathbf{L}_i}{[R^{(1)}(r)]^2}. \tag{2.22}$$

Averaging over the $\mathbf{L}_i$, one obtains

$$\begin{aligned}-\zeta_q^{-1}&=\lim_{r\to\infty}\frac{1}{qr}\ln\left[\overline{\left(\frac{\prod_{i=1}^{r}\det \mathbf{L}_i}{[R^{(1)}(r)]^2}\right)^q}\right]\\&=\lim_{r\to\infty}\frac{1}{qr}\ln\left[\overline{\left(\frac{\prod_{i=1}^{r}(2\sinh\beta J_i)}{[R^{(1)}(r)]^2}\right)^q}\right].\end{aligned} \tag{2.23}$$

In this way we have numerically studied several cases with both random couplings and/or random magnetic fields. For fixed couplings and random fields two distributions were considered:

(i) Gaussian of zero mean and variance σ^2;

(ii) uniform on an interval $[-h_0, h_0]$.

To get an idea of the role played by the fluctuations of the free energy against that of the spin–spin correlation function, for $J/kT=2$ and $h_0/kT=0.5$ we find $\lambda_1\approx 2.048$

and $L(-2)/(-2) \approx 2.037$ while the correlation decay is given by $\zeta_0^{-1} = \lambda_1 - \lambda_2 \approx 0.094$ and $\zeta_1^{-1} \approx 0.075$. This provides clear evidence that a small deviation of $L(q)/q$ from λ_1 (i.e. small free energy fluctuations) can induce a substantial difference between the most probable correlation length ζ_0 and the average one ζ_1.

We have also considered the case of random couplings and random fields. Here there is no clear-cut relation, like (2.19), between the ζ_q and the $L(q)$; a numerical computation remains, nevertheless, equally straightforward. We have considered in particular the case of couplings uniform on the interval $[J_1, J_2]$ in the presence of Gaussian fields of zero mean and variance σ.

Let us mention that it is possible to extend our argument to higher dimensions, by considering 2D strips and 3D bars, as we shall briefly sketch in section 4.

3. Numerical simulations

In this section we shall confront the results obtained via the transfer matrix formalism with direct Monte Carlo simulations of the model (2.1). In our Monte Carlo calculations the generalised correlation functions (2.14) are defined as

$$C_q(r) = \left(\frac{1}{N} \sum_{i=1}^{N} |G_i(r)|^q \right)^{1/q} \tag{3.1}$$

where the thermal average inside $G_i(r) = \langle \sigma_i \sigma_{i+r} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+r} \rangle$ is given by the average over the Monte Carlo configurations and the disorder average by a spatial average. We have used periodic boundary conditions $\sigma_{N+i} = \sigma_i$. The typical value of N, the number of spins in the chain, used in the simulations is 12 000, whereas the number of Monte Carlo configurations is 20 000.

The characteristic lengths ζ_q are obtained directly from their definition

$$C_q(r) \sim \exp(-r/\zeta_q). \tag{3.2}$$

As an example, in figure 1 is shown $\ln C_q(r)$ plotted as a function of r with $q = 1$ (full circle), 3 (full box) for βJ_i uniform on the interval [0.2, 1.2] and βh_i Gaussian of zero mean and variance $\sigma = 0.1$. The broken straight lines represent the $C_q(r)$ obtained with ζ_q derived from the fluctuations of the Lyapunov exponents of the product of random transfer matrices. The full line has slope $-\zeta_0^{-1}$. The agreement is excellent, as one can see from the figure.

Another example is shown in figure 2, where $\ln C_q(r)$ is plotted as a function of r for $q = 1$ (full circle), $q = 3$ (full box) taking $\beta J_i = 2$ and random βh_i uniformly distributed in the interval $[-0.5, 0.5]$. The agreement between the Monte Carlo simulations and the transfer matrices approach is, also in this case, excellent.

We have finally studied the behaviour of the most probable value as a function of r. This is done by computing the probability distribution of $G_i(r)$ for different values of r; an example is shown in figure 3. As expected, for increasing r, it becomes more and more peaked about a value that tends to zero. This value is identified as the most probable value of $G_i(r)$. If one plots the logarithm of the most probable value as a function of r one finds that this scales like $(\lambda_2 - \lambda_1)r$. See figure 1, where the crosses denotes the logarithm of the most probable value. The full line has slope $\lambda_2 - \lambda_1$. Notice that the characteristic length of the most probable value is found to be different from that of the average correlation function $C_1(r)$ (cf figures 1 and 2).

The non-trivial behaviour of ζ_q as function of q indicates that the leading contribution to the correlation functions $C_q(r)$ must come from very rare events, i.e. large

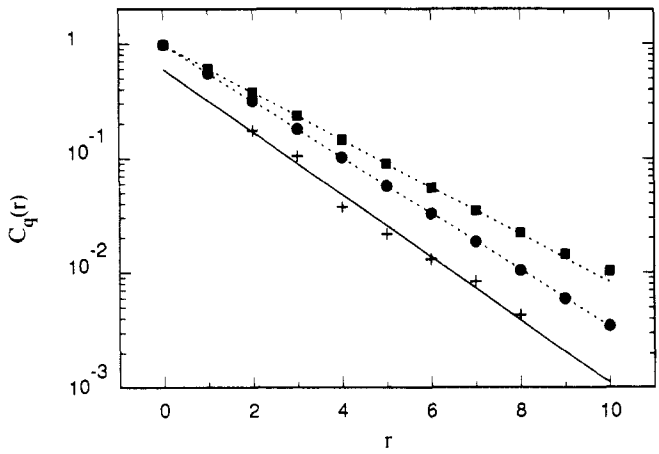

Figure 1. The $C_q(r)$ as a function of r for βJ_i uniform on the interval [0.2, 1.2] and βh_i Gaussian of zero mean and variance $\sigma = 0.1$. The symbols refer to $q = 1$ (full disc) and $q = 3$ (full square). The broken straight lines are the $C_q(r)$ predicted by the fluctuations of the Lyapunov exponents. The numerical errors are comparable with the symbol size. The '+' refer to the most probable value obtained from histograms of $G_i(r)$. The full line has slope $-\zeta_0^{-1}$.

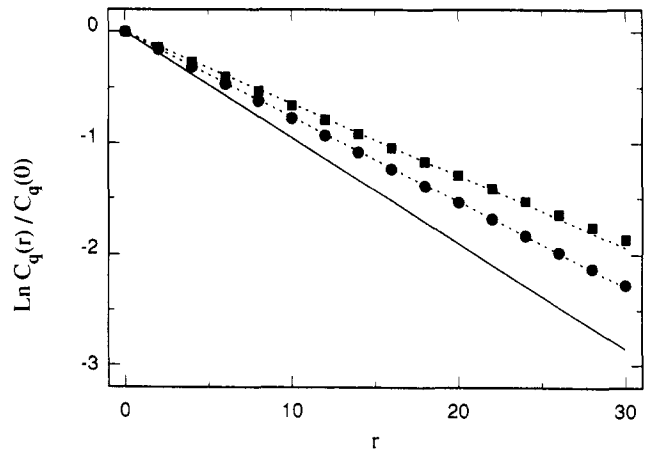

Figure 2. The $C_q(r)$ as a function of r for $\beta J_i = 2$ and βh_i uniform on the interval $[-0.5, 0.5]$. The symbols refer to $q = 1$ (full disc) and $q = 3$ (full square). The broken straight lines are the $C_q(r)$ predicted by the fluctuations of the Lyapunov exponents. The numerical errors are comparable with the symbol size. The full line has slope $-\zeta_0^{-1}$ and gives the decay of the most probable value.

fluctuations. Therefore we could not limit ourselves to look only at ζ_1, since the whole hierarchy of exponents ζ_q arises as the natural experimental characterisation of correlation functions.

From a practical point of view, it is very important to discuss the effect of a finite number of spins N in numerical experiments. A quantitative characterisation of the finite-size fluctuations can be achieved by looking at the variance

$$\frac{\overline{g^2(r)} - \overline{g(r)}^2}{\overline{g(r)}^2} \sim \frac{1}{N} \exp[-2(\zeta_2^{-1} - \zeta_1^{-1})r] \tag{3.3}$$

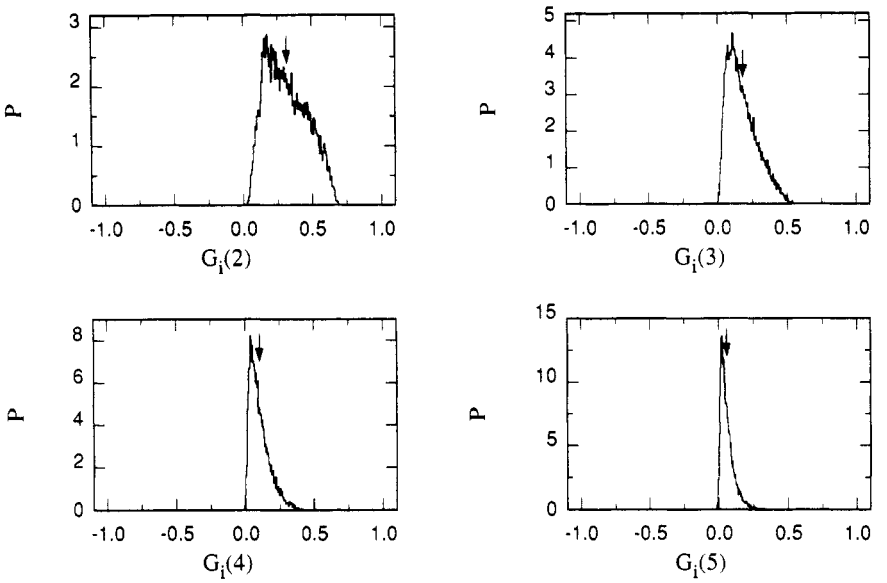

Figure 3. The probability distribution P of $G_i(r)$ plotted against $G_i(r)$ ($r = 2, 3, 4, 5$) for the case of figure 1. The arrows indicate the mean value of $G(r)$.

of the cumulant two-point function

$$g(r) = \frac{1}{N} \sum_{i=1}^{N} (\langle \sigma_i \sigma_{i+r} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+r} \rangle). \tag{3.4}$$

This means that in a numerical study the fluctuations of $g(r)$ can be observed for $r \leqslant c \ln N$, with $c^{-1} = 2(\zeta_1^{-1} - \zeta_2^{-1})$. As $\ln N$ is quite a small number even in large systems (for us $\ln N \approx 10$), in the literature it has been argued that it is practically impossible to measure ζ_1, almost all data being concentrated around the most probable value [5]. This is not correct since the constant c need not be (and is not) small, so that $c \ln N$ may be sizeable. For $r \geqslant c \ln N$, one should expect to observe for almost all realisations the most probable correlation decay (given by ζ_0) because of the Oseledec theorem. This is, however, a quite academic statement: in this range of r the values of $g(r)$ are so small that they cannot be measured, due of the errors of the statistics in a Monte Carlo calculation. Equally severe, perhaps worse, limitations, due to finite precision of instruments, also occur in real experiments. This answers, in a positive sense, the point raised in reference [5] as to the feasibility of detecting the average value of the spin-spin correlation function (and of the generalised correlation functions) in a Monte Carlo simulation.

4. Conclusions

In order to describe the sample-to-sample fluctuations of two-point correlation functions in random systems, we study a set of generalised exponents ζ_q. The ζ_q are related to the finite-N fluctuations in the ratio of the largest eigenvalues of the product of N random transfer matrices. They generalise the difference of the first two Lyapunov exponents. For the specific problem of spin-spin correlations in one-dimensional

nearest-neighbour random magnets, we found excellent agreement with direct Monte Carlo simulations.

We also discussed the crossover between the average value of the correlation decay and its most probable value, as well as the effect of finite size and of finite statistics in Monte Carlo calculations.

We will conclude with some comments on calculations in higher dimensions: in the context of the transfer matrix formalism we must consider strips of size l and so $2^l \times 2^l$ matrices for two dimensions (or bars of size l^2 and $2^{l^2} \times 2^{l^2}$ matrices, for three dimensions). The typical correlation decay is thus given by the sum of exponentials related to the differences of the kth Lyapunov exponent ($k = 2, 3, \ldots, 2^l$) from the first Lyapunov exponent [10]. However, for distances large enough, $\gamma_2 - \gamma_1$ gives the leading contribution—the corrections are exponentially small if there are no degeneracy problems, i.e. phase-transition-like phenomena. One could therefore repeat the arguments of our paper, taking into account this warning. Nevertheless, equation (2.22) *cannot* be used since the determinant of the transfer matrix is now related to the sum of all the 2^l Lyapunov exponents. We must therefore introduce the effective rate of surface increase under N iterations of the transfer matrices $\mathbf{L}$,

$$R^{(2)}(N) = \frac{\|\boldsymbol{\xi}_1(N) \times \boldsymbol{\xi}_2(N)\|}{\|\boldsymbol{\xi}_1(0) \times \boldsymbol{\xi}_2(0)\|} \sim \exp\{N[\gamma_1(N) + \gamma_2(N)]\} \tag{4.1}$$

where $\times$ indicates the external product, $\boldsymbol{\xi}_1(0)$ and $\boldsymbol{\xi}_2(0)$ are generic non-parallel vectors. Indeed, the standard procedure for numerical computations of the first two Lyapunov exponents consists in looking at the typical increase rate of $R^{(2)}$ [11], while the increase rate of its order-q moments leads one to define a set of generalised exponents $L^{(2)}(q)$ [12]. We must recall these somewhat technical definitions since they allow us to compute the generalised correlation lengths in any disordered model which can be studied by transfer matrices by using

$$-\zeta_q^{-1} = \lim_{r \to \infty} \frac{1}{qr} \ln \left[\overline{\left(\frac{R^{(2)}(r)}{[R^{(1)}(r)]^2} \right)^q} \right]. \tag{4.2}$$

This extension of equation (2.22) paves the way to a full characterisation of correlation functions for nearest-neighbour disordered models in higher dimensions. Indeed it is known [13] that for product of very large random matrices the fluctuations of the effective Lyapunov exponent are not negligible. We, therefore, expect a non-trivial behaviour of ζ_q as function of q.

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